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Products and Convolutions of Gaussian Distributions

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Abstract

It is well known that the product and the convolution of two Gaussian distributions are also Gaussian distributions. This memo provides derivations for the mean and standard deviation of the resulting Gaussian distributions in both cases. These results are useful in calculating the effects of smoothing applied as an intermediate step in various algorithms.

1 The Product of Two Gaussian Distributions

We wish to find the product of two Gaussian distributions

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_f} e^{-\frac{(x-\mu_f)^2}{2\sigma_f^2}} \quad \text{and} \quad g(x) = \frac{1}{\sqrt{2\pi}\sigma_g} e^{-\frac{(x-\mu_g)^2}{2\sigma_g^2}} \tag{1}$$

in the most general case i.e. non-identical means. The product gives

$$f(x)g(x) = \frac{1}{2\pi\sigma_f\sigma_g} e^{-\left(\frac{(x-\mu_f)^2}{2\sigma_f^2} + \frac{(x-\mu_g)^2}{2\sigma_g^2}\right)} \tag{2}$$

Examining the term in the exponent

$$\alpha = \frac{(x-\mu_f)^2}{2\sigma_f^2} + \frac{(x-\mu_g)^2}{2\sigma_g^2} \tag{3}$$

we can expand the two quadratics and collect terms in powers of x to give

$$\alpha = \frac{(\sigma_f^2 + \sigma_g^2)x^2 - 2(\mu_f\sigma_g^2 + \mu_g\sigma_f^2)x + \mu_f^2\sigma_g^2 + \mu_g^2\sigma_f^2}{2\sigma_f^2\sigma_g^2} \tag{4}$$

Dividing through by the coefficient of x^2 gives

$$\alpha = \frac{x^2 - 2\frac{\mu_f\sigma_g^2 + \mu_g\sigma_f^2}{\sigma_f^2 + \sigma_g^2}x + \frac{\mu_f^2\sigma_g^2 + \mu_g^2\sigma_f^2}{\sigma_f^2 + \sigma_g^2}}{2\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2 + \sigma_g^2}} \tag{5}$$

This is again a quadratic, and so Eq. 2 is a Gaussian. Comparing the terms in Eq. 6 to a the usual Gaussian form

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x^2 - 2\mu x + \mu^2)}{2\sigma^2}} \tag{6}$$

we can identify the standard deviation of the product as the square root of half of the denominator of Eq. 5 and the mean as half the coefficient of x i.e.

$$\sigma_{fg} = \sqrt{\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2 + \sigma_g^2}} \quad \text{and} \quad \mu_{fg} = \frac{\mu_f\sigma_g^2 + \mu_g\sigma_f^2}{\sigma_f^2 + \sigma_g^2} \tag{7}$$

We can now either write down the product $f(x)g(x)$ in the usual Gaussian form directly, or proceed from Eq. 5. Taking the latter route, suppose that γ is the factor required to complete the square in α i.e.

$$\alpha = \frac{(x-\mu_{fg})^2}{2\sigma_{fg}^2} - \gamma \tag{8}$$

Substituting back into Eq. 2 gives

$$f(x)g(x) = \frac{e^{-\gamma}}{2\pi\sigma_f\sigma_g} e^{-\frac{(x-\mu_{fg})^2}{2\sigma_{fg}^2}} \tag{9}$$

The normalisation factor can be found by integrating over all x using the standard result [2]

$$\int_{-\infty}^{\infty} e^{-ax} = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{\infty} \frac{e^{-\gamma}}{2\pi\sigma_f\sigma_g} e^{-\frac{(x-\mu_{fg})^2}{2\sigma_{fg}^2}} = \frac{e^{-\gamma}}{\sqrt{2\pi(\sigma_f^2+\sigma_g^2)}} \tag{10}$$

and so the final Gaussian form for the product is

$$P_{fg}(x) = \frac{1}{\sqrt{2\pi}\sigma_{fg}} e^{-\frac{(x-\mu_{fg})^2}{2\sigma_{fg}^2}}$$

2 The Convolution of Two Gaussian Distributions

We wish to find the convolution of two Gaussian distributions

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_f} e^{-\frac{(x-\mu_f)^2}{2\sigma_f^2}} \quad \text{and} \quad g(x) = \frac{1}{\sqrt{2\pi}\sigma_g} e^{-\frac{(x-\mu_g)^2}{2\sigma_g^2}} \tag{11}$$

in the most general case i.e. non-identical means. The convolution of two functions $f(t)$ and $g(t)$ over a finite range[1] is defined as

$$\int_0^x f(x-\tau)g(\tau)d\tau = f \otimes g \tag{12}$$

However, the usual approach is to use the convolution theorem [3],

$$F^{-1}[F(f(x))F(g(x))] = f(x) \otimes g(x) \tag{13}$$

where F is the Fourier transform

$$F(f(x)) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}\ dx \tag{14}$$

and F^{-1} is the inverse Fourier transform

$$F^{-1}(F(k)) = \int_{-\infty}^{\infty} F(k)e^{2\pi ikx}\ dk \tag{15}$$

Using the transformation

$$x' = x - \mu_f \tag{16}$$

the Fourier transform of $f(x)$ is given by

$$F(f(x)) = \frac{1}{\sqrt{2\pi}\sigma_f} \int_{-\infty}^{\infty} e^{-\frac{x'^2}{2\sigma_f^2}} e^{-2\pi ik(x'-\mu_f)}\ dx' = \frac{e^{-2\pi ik\mu_f}}{\sqrt{2\pi}\sigma_f} \int_{-\infty}^{\infty} e^{-\frac{x'^2}{2\sigma_f^2}} e^{-2\pi ikx'}\ dx' \tag{17}$$

Using Euler's formula [3],

$$e^{-i\theta} = \cos\theta - i\sin\theta \tag{18}$$

we can split the term in $e^{x'}$ to give

$$F(f(x)) = \frac{e^{-2\pi ik\mu_f}}{\sqrt{2\pi}\sigma_f} \int_{-\infty}^{\infty} e^{-\frac{x'^2}{2\sigma_f^2}} [\cos(2\pi kx') - i\sin(2\pi kx')]\ dx' \tag{19}$$

[1]In practice, convolutions are more often performed over an infinite range

$$\int_{-\infty}^{\infty} f(x-\tau)g(\tau)d\tau = f \otimes g$$

The term in $\sin(x')$ is odd and so its integral over all space will be zero, leaving

$$F(f(x)) = \frac{e^{-2\pi ik\mu_f}}{\sqrt{2\pi}\sigma_f} \int_{-\infty}^{\infty} e^{-\frac{x'^2}{2\sigma_f^2}} \cos(2\pi kx')\ dx' \tag{20}$$

This integral is given in standard form in [1]

$$\int_0^{\infty} e^{-at^2} \cos(2xt)\ dt = \frac{1}{2}\sqrt{\frac{\pi}{a}} e^{-\frac{x^2}{a}} \tag{21}$$

and so

$$F(f(x)) = e^{-2\pi ik\mu_f} e^{-2\pi^2\sigma_f^2 k^2} \tag{22}$$

The second term in this expression is a Gaussian distribution in k: the Fourier transform of a Gaussian distribution is another Gaussian distrbution. The first term is a phase term accounting for the mean of $f(x)$ i.e. its offset from zero. The Fourier transform of $g(x)$ will give a similar expression, and so

$$F(f(x))F(g(x)) = e^{-2\pi ik\mu_f} e^{-2\pi^2\sigma_f^2 k^2} e^{-2\pi ik\mu_g} e^{-2\pi^2\sigma_g^2 k^2} = e^{-2\pi ik(\mu_f+\mu_g)} e^{-2\pi^2(\sigma_f^2+\sigma_g^2)k^2} \tag{23}$$

Comparing Eq. 23 to Eq. 22, we can see that it is the Fourier transform of a Gaussian distribution with mean and standard deviation

$$\mu_{f\otimes g} = \mu_f + \mu_g \quad \text{and} \quad \sigma_{f\otimes g} = \sqrt{\sigma_f^2 + \sigma_g^2} \tag{24}$$

and therefore, since the Fourier transform is invertible,

$$P_{f\otimes g}(x) = F^{-1}[F(f(x))F(g(x))] = \frac{1}{\sqrt{2\pi(\sigma_f^2+\sigma_g^2)}} e^{-\frac{(x-(\mu_f+\mu_g))^2}{2(\sigma_f^2+\sigma_g^2)}} \tag{25}$$

3 Summary

It is well known that the product and the convolution of a pair of Gaussian distributions are also Gaussian distributions. However, the derivations are not commonly seen in the literature, particularly in the case of Gaussian distributions with non-identical means. This document has provided both derivations. In the case of the product of two Gaussian distributions, the result is a Gaussian distribution with mean and standard deviation

$$\mu_{fg} = \frac{\mu_f\sigma_g^2 + \mu_g\sigma_f^2}{\sigma_f^2 + \sigma_g^2} \quad \text{and} \quad \sigma_{fg} = \sqrt{\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2 + \sigma_g^2}}$$

where μ_f and μ_g are the means of the two original Gaussians and σ_f and σ_g are their standard deviations. In the case of the convolution of two Gaussian distributions, the result is again a Gaussian distribution with mean and standard deviation

$$\mu_{f\otimes g} = \mu_f + \mu_g \quad \text{and} \quad \sigma_{f\otimes g} = \sqrt{\sigma_f^2 + \sigma_g^2}$$

These results, particularly the second result, can be useful in calculating the effects of Gaussian smoothing applied as an intermadiate step in various machine vision algorithms.

References

[1] M Abramowitz and I A Stegun. *Handbook of Mathematical Functions.* National Bureau of Standards, Washington DC, 1972.

[2] R J Barlow. *Statistics: a guide to the use of statistical methods in the physical sciences.* John Wiley and Sons Ltd., 1989.

[3] M L Boas. *Mathematical Methods in the Physical Sciences.* John Wiley and Sons Ltd., 1983.